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Series expansions for lattice Green functions

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Abstract. Lattice Green functions appear in lattice gauge theories, in lattice models of statistical physics and in random walks. Here, space coordinates are treated as parameters and series expansions in the mass are obtained. The singular points in arbitrary dimensions are found. For odd dimensions these are branch points with half odd-integer exponents, while for even dimensions they are of the logarithmic type. The differential equations for one, two and three dimensions are derived, and the general form for arbitrary dimensions is indicated. Explicit series expressions are found in one and two dimensions. These series are hypergeometric functions. In three and higher dimensions the series are more complicated. Finally an algorithmic method by Vohwinkel, Lüscher and Weisz is shown to generalize to arbitrary anisotropies and mass.

1. Introduction

Green functions, also known as two-point functions or propagators, are ubiquitous in physics. They appear in classical field theories, quantum mechanics and quantum field theories to name but a few areas. When a discrete space–time approach is used as a regulator, the continuum Green functions become lattice functions. In the problems of condensed matter physics, both the continuum and lattice approach are natural. Random walks on a lattice provide another field of application for Green functions, where they are generating functions for the return probability of a random walker on a lattice [1]. Given an isotropic random walk occurring in steps of one unit on a hypercubic lattice, one considers the probability of visiting a given site, or of returning to the starting site, after n steps. This problem was first considered by Pólya [2] who showed that this probability tends to one only in one and two dimensions, for n tending to infinity. This is the recurrence/transience transition which is an important result in the study of random walks.

The free propagator appears naturally in the perturbative expansion of field theories. A lattice provides a regulator for the infinities in such expansions, and the four-dimensional discrete Green functions appear naturally [3–5]. The three-dimensional function appears in the study of effective three-dimensional theories [6]. A lattice regularization has also been used in [7] to study non-relativistic quantum scattering in two and three space dimensions. For dimensions greater than one, the continuum free Green function is singular at the origin, while the lattice version is finite. This provides a regularization of the UV divergences. In yet another context, the propagator was used to find the resistance of a network of resistors [8]. (See also [9] in relation to conformal invariance.)

Although the integral representation of the lattice Green function for the hypercubic lattice is well known, the analytic calculation of this integral remains a challenge. A closed form

appears to exist only in one dimension and for a special case in two dimensions. In higher dimensions there are several partial results [1, 10]. A rather recent numerical approach was used in [11] and developed in [12] for the massless free propagator. It was applied to the two-dimensional case in [13]. Similar position space methods were considered earlier in [14], and later in [15].

In this paper the free propagator as a function of the mass squared which is allowed to vary in the whole complex plane is considered. The space coordinates are treated as parameters. Series expansions are first obtained for large values of the mass. In one and two dimensions these series are hypergeometric while in higher dimensions they are not. This large mass expansion follows from the multi-dimensional integral representation of the lattice Green function. However, this representation does not allow one to find the series expansion around other values of the mass. To this end recurrence relations and corresponding differential equations in the mass squared are derived. This approach gives the singularities of the Green function and allows one to expand around any point. Non-trivial monodromies are found around the singular points.

The paper is organized as follows. Section 2 introduces the notation and the well known integral representation for the minimal anisotropic lattice Green function for the simple cubic lattice in d dimensions. Some general properties are also discussed. The one-dimensional integral representation in terms of Bessel functions is given. A simple derivation for the location of the singularity in the complex mass plane is found. A general expansion of the lattice Green function in the inverse of the mass is obtained, and a recurrence relation between dimensions for the coefficients of this expansion is noted. In section 3 the one-dimensional case is studied in detail as a simple illustration of the methods used. This also serves as a guide to the features common to all dimensions. The same methods are applied to the two-dimensional Green function and two novel expansions are found (section 4). The recurrence relation and the differential equation are then obtained for the three-dimensional case (section 5). Section 6 concludes with general remarks for four and higher dimensions. Appendix A contains a set of formulae used in the text. In appendix B, the method of [12] is extended to arbitrary mass and anisotropies.

2. General results

Consider the hypercubic lattice in d dimensions, with unit vectors \hat{e}_j , $j = 1, \dots, d$. The discrete anisotropic Green equation is

$$Hf(\vec{x}) \equiv \sum_{j=1}^d \alpha_j [f(\vec{x} + \hat{e}_j) + f(\vec{x} - \hat{e}_j)] = 2\beta f(\vec{x}) + \delta_{\vec{x}, \vec{0}}. \quad (1)$$

The integers x_j label the lattice sites, and the anisotropies α_j are arbitrary but non-vanishing complex numbers. When the α_j s are omitted, they should be assumed to be all equal to 1. The action of the discrete anisotropic d -dimensional Laplace operator Δ on f is given by $(H - 2 \sum_{j=1}^d \alpha_j) f(\vec{x})$. Here the contribution of the latter term has been absorbed in β which is, depending on the context, the mass squared, the energy eigenvalue or a formal expansion parameter for a generating function. This definition of β gives a natural parity symmetry (6), and renders the sets of singularities symmetric with respect to the origin. The free massless scalar propagator of lattice gauge theories corresponds to the isotropic case, $\alpha_j = 1$ for all j , and $\beta = d$. A completely isotropic solution of the isotropic equation satisfies

$$2df(\hat{e}_j) = 2\beta f(\vec{0}) + 1 \quad j = 1, \dots, d. \quad (2)$$

Adding to a solution of (1) any solution of the homogeneous equation

$$\sum_{j=1}^d \alpha_j [f(\vec{x} + \hat{e}_j) + f(\vec{x} - \hat{e}_j)] = 2\beta f(\vec{x}) \tag{3}$$

yields another Green function. A large class of such solutions can be written as

$$h(\vec{x}|\vec{\alpha}, \beta) = \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} \frac{d^d \vec{q}}{(2\pi)^d} \exp(i\vec{q} \cdot \vec{x}) \tilde{h}(\vec{q}) \delta\left(\sum_{j=1}^d \alpha_j \cos q_j - \beta\right) \tag{4}$$

where the function \tilde{h} is arbitrary but well behaved.

One can define a *decoupling point* at $\beta = 0$, for which equation (1) becomes an equation for two sub-lattices: a lattice where $\sum_{j=1}^d x_j$ is even and one where $\sum_{j=1}^d x_j$ is odd.

The minimal solution of (1) is given by

$$G_{\pm}^{(d)}(\vec{x}|\vec{\alpha}, \beta) = \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} \frac{d^d \vec{q}}{(2\pi)^d} \frac{\exp(i\vec{q} \cdot \vec{x})}{2 \sum_{j=1}^d \alpha_j \cos q_j - 2\beta \pm i\epsilon} \tag{5}$$

where $\epsilon \rightarrow 0^+$. The ϵ prescription removes integration ambiguities at the poles.

When only d' anisotropy parameters are equal to each other, the solution (5) is invariant under the $2^{d'} \cdot d'!$ parity transformations and permutations of the x_j s. For $d' = d$, the symmetry is that of the d -dimensional hypercubic group, and the functions $G_{\pm}^{(d)}$ are completely symmetric in the absolute values of their arguments x_j . There is also a parity symmetry relating β to $-\beta$:

$$G_{\pm}^{(d)}(\vec{x}|\vec{\alpha}, -\beta) = e^{i\pi(1+X)} G_{\mp}^{(d)}(\vec{x}|\vec{\alpha}, \beta) \tag{6}$$

where $X \equiv \sum_{j=1}^d |x_j|$, and a complex conjugation symmetry

$$(G_{\pm}^{(d)}(\vec{x}|\vec{\alpha}, \beta))^* = G_{\mp}^{(d)}(\vec{x}|\vec{\alpha}^*, \beta^*) \tag{7}$$

The latter symmetry shows that the two Green functions are complex conjugates of each other.

The exponentiation formula

$$\frac{n!}{A^{n+1}} = \int_0^{\infty} dt t^n \exp(-tA) \quad \text{Re}(A) > 0 \quad n = 0, 1, 2, \dots \tag{8}$$

and the integral representations (74) for the Bessel functions yield a one-dimensional integral representation for (5)

$$G_{\pm}^{(d)}(\vec{x}|\vec{\alpha}, \beta) = -\frac{(\pm i)^{1+X}}{2} \int_0^{+\infty} dt \exp\left(-\frac{t\epsilon}{2} \mp it\beta\right) J_{|x_1|}(\alpha_1 t), \dots, J_{|x_d|}(\alpha_d t). \tag{9}$$

In this expression ϵ can be set to zero. For $\alpha_j \in \mathbb{R}$, the integrals converge in the domains $\mp \text{Im} \beta \geq 0$, without a finite number of real points. Note that (9) satisfies (1) by virtue of properties (78)–(81).

To find the singularities of $G_{\pm}^{(d)}$ in β , we consider first the initial integral representation (5). For $|\beta| > \sum_j |\alpha_j|$, the integrand is a continuous function, without singularities, integrated over a compact domain. Thus, the Green function has no singularities for these values of β . This includes the point at infinity. Now we consider, for simplicity, the case without anisotropies ($\alpha_j = 1$). Equation (5) implies that the possible singularities are real. When $d = 1$, the integral (9) converges provided the oscillating cosine of the asymptotic expansion (76) is not ‘cancelled’ by $\exp(\mp it\beta)$. For $\beta = \pm 1$, and only for these values, the integral has a diverging contribution of the form $\int^{\infty} dt/\sqrt{t}$, which results in the branch points $\beta = \pm 1$. This is confirmed by the explicit expressions given in section 3. The same reasoning holds for $d = 2$. The product of the two cosines yields one divergent contribution, $\int dt/t$ for three values of β :

0 and ± 2 . This is confirmed in section 4. For $\beta = 0$, note that this approach also predicts a lack of divergence for points on the odd sub-lattice. One has $\beta \ln \beta$, which vanishes as β tends to 0. For $d \geq 3$, there are enough powers of \sqrt{t} to give a converging integral at all β . This corresponds to the recurrence/transience transition in the context of random walks. Let $[n]$ denote the integer part of n . Taking $[(d-1)/2]$ β -derivatives of (9), and using the asymptotic expansion for the Bessel function, yields $d+1$ singularities: $\beta_0 = -d, -d+2, \dots, d-2, d$. For $d \geq 1$ and odd, they are of the branch point type $(\beta - \beta_0)^{p-\frac{1}{2}}$, where p is a non-negative integer. For $d \geq 2$ and even, the singularities are logarithmic of the type $(\beta - \beta_0)^{p'} \ln^{q'}(\beta - \beta_0)$, where p' is a non-negative integer and q' a positive integer. A slightly more complicated but essentially similar analysis applies for arbitrary anisotropies. The location of the singularities will depend explicitly on the α_j s. These results can be made rigorous through the use of a Tauberian theorem.

The ϵ prescription becomes a convergence factor when one uses the series expansions (75) of the Bessel functions. This gives

$$G_{\pm}^{(d)}(\vec{x}|\vec{\alpha}, \beta) = -\frac{1}{2\beta^{1+X}} \sum_{n=0}^{\infty} c_{X+2n}^{(d)}(\vec{x}|\vec{\alpha}) \beta^{-2n} \quad (10)$$

where

$$c_n^{(d)}(\vec{x}|\vec{\alpha}) \equiv \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} \frac{d^d \vec{q}}{(2\pi)^d} \exp(i\vec{q} \cdot \vec{x}) \left(\sum_{j=1}^d \alpha_j \cos q_j \right)^n \quad (11)$$

and

$$c_n^{(d)}(\vec{x}|\vec{\alpha}) = 0 \quad n = 0, \dots, X-1 \quad (12)$$

$$c_{X+2n}^{(d)}(\vec{x}|\vec{\alpha}) = (X+2n)! \prod_{j=1}^d \left(\frac{\alpha_j}{2}\right)^{|x_j|} \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \prod_{j=1}^d \left(\frac{\alpha_j}{2}\right)^{2k_j} \times \frac{\delta_{k_1+\dots+k_d, n}}{k_1! \cdots k_d! (|x_1|+k_1)! \cdots (|x_d|+k_d)!} \quad n \geq 0. \quad (13)$$

In particular one finds

$$G_{\pm}^{(d)}(\vec{x}|\vec{\alpha}, \beta) \sim -\frac{X! \prod_{j=1}^d \alpha_j^{|x_j|}}{|x_1|! \cdots |x_d|! (2\beta)^{1+X}} \quad \beta \rightarrow \infty \quad (14)$$

which shows the Green function to vanish faster than the simple estimate $1/(2\beta)$ obtained from the integral representation (5). A more compact form of the coefficients (13) can be obtained by the pairwise replacement of Bessel functions through identity (82). This is done for $d = 2, 3, 4$ in the following sections.

The result (10) does not depend on the sign of the ϵ prescription. This is easily understood by noticing that for $|\beta| > \sum_{j=1}^d |\alpha_j|$ the denominator in (5) does not have poles and therefore ϵ can be set to zero. The large β expansion then yields (10) with the c_n coefficients given by (11). One can also conclude that the expansion (10)–(13) converges at least for $|\beta| > \sum_{j=1}^d |\alpha_j|$. Convergence at the generalized massless point ($\beta = \sum_{j=1}^d |\alpha_j|$) depends on the dimensionality of the lattice. This is related to the recurrence/transience of the random walk. In one and two dimensions the series diverge at this point, despite the ϵ prescription. In higher dimensions the series converge.

The random walk interpretation of the c_n s is the following. For a random walker starting from the origin, let $P_n(\vec{x})$ be the probability of visiting the site \vec{x} after n unit steps on the d -dimensional hypercubic lattice. We take the anisotropies to be positive and such that $\sum_{j=1}^d \alpha_j = d$. The probability of jumping from \vec{x} to $\vec{x} + \hat{e}_j$ or to $\vec{x} - \hat{e}_j$ is $\frac{\alpha_j}{2d}$. One then

has $P_n(\vec{x}) = \frac{1}{d^n} c_n^{(d)}(\vec{x}|\vec{\alpha})$. The vanishing of c_n for $n < X$ is therefore natural since X is the minimal number of steps required to reach point \vec{x} .

General relations between the coefficients c_n for different dimensions can be simply found. Let d' be any positive integer smaller than d . Expanding $\sum_{j=1}^d \alpha_j \cos q_j$ using the binomial formula readily yields

$$c_n^{(d)}(\vec{x}|\vec{\alpha}) = \sum_{k=0}^n \binom{n}{k} c_k^{(d')} (x_1, \dots, x_{d'}|\alpha_1, \dots, \alpha_{d'}) c_{n-k}^{(d-d')} (x_{d'+1}, \dots, x_d|\alpha_{d'+1}, \dots, \alpha_d). \tag{15}$$

Note that these relations are valid for arbitrary anisotropies. One can also get other equations by expanding with the multinomial formulae.

3. The one-dimensional Green function

In one dimension it is possible to obtain closed form expressions, and corresponding series expansions. The anisotropy parameter α_1 is set to one as it is an irrelevant overall factor.

The closed forms are obtained by integration in the complex q -plane. The integration contour is the rectangular path $]-\pi + i\infty, -\pi] \cup [-\pi, \pi] \cup [\pi, \pi + i\infty[\cup]\pi + i\infty, -\pi + i\infty[$, for the upper half-plane, and its reflection about the real axis for the lower half-plane. The part at infinity gives a vanishing contribution, while the two side contributions cancel each other because $x \equiv x_1$ is an integer. One finds

$$G_{\pm}^{(1)}(x|\beta) = \pm \frac{e^{\pm ik|x|}}{2i \sin k} \quad \beta = \cos k \quad k \in]0, \pi[. \tag{16}$$

One can compare (16) to its continuum counterpart

$$g_{\pm}^{(1)}(x|k) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{\exp(iqx)}{k^2 - q^2 \pm i\epsilon} = \pm \frac{e^{\pm ik|x|}}{2ik}. \tag{17}$$

One also finds

$$G_{+}^{(1)}(x|\beta) = G_{-}^{(1)}(x|\beta) = \begin{cases} + \frac{e^{ik|x|}}{2i \sin k} & k_1 \in [0, \pi] \quad k_2 > 0 \\ - \frac{e^{-ik|x|}}{2i \sin k} & k_1 \in]0, \pi[\quad k_2 < 0 \end{cases} \tag{18}$$

where $\beta = \cos(k_1 + ik_2)$ and $k = k_1 + ik_2$. Note that the positive exponential $\frac{e^{+k_2|x|}}{2 \sinh k_2}$ ($k_2 > 0$), despite satisfying equation (1), is not obtained. This was to be expected from the integral representation (5), since for $|\beta| > 1$ the integrand has no singular point and the integral must vanish as $|x| \rightarrow \infty$.

The general expression (10)–(13) reduces to

$$\begin{aligned} G_{\pm}^{(1)}(x|\beta) &= -\frac{1}{(2\beta)^{|x|+1}} \sum_{k=0}^{\infty} \frac{(|x| + 2k)!}{k!(|x| + k)!} (2\beta)^{-2k} \\ &= -\frac{1}{(2\beta)^{|x|+1}} {}_2F_1\left(\frac{|x|}{2} + \frac{1}{2}, \frac{|x|}{2} + 1; |x| + 1; \beta^{-2}\right). \end{aligned} \tag{19}$$

This series converges uniformly for $|\beta| > 1$ and simply for $|\beta| = 1, \beta \neq \pm 1$; it diverges at $\beta = \pm 1$ and for $|\beta| < 1$. The equality of the right-hand sides of (18) and (19) was previously known [16]. The recurrence relation for the coefficients of the Green function, with $c_n \equiv c_n^{(1)}(x|1)$, reads

$$(n^2 - x^2)c_n - n(n - 1)c_{n-2} = 0. \tag{20}$$

The function $G_{\pm}^{(1)}$ satisfies a second-order differential equation in β , of the hypergeometric type:

$$(\beta^2 - 1)y'' + 3\beta y' + (1 - x^2)y = 0. \tag{21}$$

The indices at the three regular singular points $\beta = \pm 1$ and ∞ are $(-\frac{1}{2}, 0)$ and $(1 - |x|, 1 + |x|)$, respectively. The two branch points $\beta = \pm 1$ imply the existence of monodromies in the complex-mass plane. Solution (19) corresponds to $(\beta = \infty; s = 1 + |x|)$. It is then natural to investigate the properties of the solution corresponding to $(\beta = \infty; s = 1 - |x|)$

$$y_1(\beta) = 2^{|x|-2} \beta^{|x|-1} {}_2F_1\left(\frac{1 - |x|}{2}, 1 - \frac{|x|}{2}; 1 - |x|; \beta^{-2}\right) \quad x \neq 0 \tag{22}$$

$$y_{11}(\beta) = \frac{1}{4\beta} \cdot \frac{1}{\sqrt{1 - \beta^{-2}}} \ln\left(\frac{1}{\beta}\right) + \frac{1}{4} \sum_{n=0}^{\infty} \beta^{-(2n+1)} \frac{d}{ds} \left(\frac{\Gamma(n + \frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(n + \frac{s+1}{2})\Gamma(\frac{s}{2})} \right)_{|s=1} \tag{23}$$

with ${}_2F_1(0, \frac{1}{2}; 0, \beta^{-2}) \equiv 1$. The derivative of the coefficients in (23) can be written using the function $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$. The appearance of the logarithm for this second solution is due to the degeneracy of the indices at $x = 0$. The hypergeometric series (22) truncates to polynomials in $\beta^{\pm 1}$. Note that this solution almost provides a Green function. The *a priori* arbitrary normalization of a solution was chosen to be equal to $2^{|x|-2}$ in (22) so that the one-dimensional Green equation, $f(x + 1|\beta) + f(x - 1|\beta) - 2\beta f(x|\beta) = \delta_{x,0}$, is satisfied by $f(x|\beta)$ given in (22) and $f(0|\beta)$ set to 0.

The expansion around $\beta = 1$ can be carried out similarly. The two solutions are

$$y_0(\beta) = {}_2F_1\left(1 - |x|, 1 + |x|; \frac{3}{2}; \frac{1 - \beta}{2}\right) \tag{24}$$

$$= \frac{\sin\left(2|x|\text{Arcsin}\left(\sqrt{\frac{1-\beta}{2}}\right)\right)}{|x|\sqrt{1 - \beta^2}} \quad \text{for } \beta \in] - 1, 1] \tag{25}$$

$$y_{1/2}(\beta) = \frac{1}{\sqrt{\beta - 1}} {}_2F_1\left(\frac{1}{2} - |x|, \frac{1}{2} + |x|; \frac{1}{2}; \frac{1 - \beta}{2}\right) \tag{26}$$

$$= -i \frac{\sqrt{2} \cos\left(2|x|\text{Arcsin}\left(\sqrt{\frac{1-\beta}{2}}\right)\right)}{\sqrt{1 - \beta^2}} \quad \text{for } \beta \in] - 1, 1[. \tag{27}$$

These solutions are valid for all values of x . In (27) the choice $\sqrt{-1} = i$ was made. The solution y_0 is regular at $\beta = 1$ and is a solution to the Green equation when appropriately normalized, while $y_{1/2}$ is singular at $\beta = 1$, and is a solution to the homogeneous equation (3). Linear combinations of these solutions provide the two analytic continuations to the function defined by (19):

$$\frac{1}{2\sqrt{2}}(\sqrt{2}|x|y_0 \pm y_{1/2}) = \pm \frac{1}{2i \sin k} e^{\pm ik|x|} \quad \beta = \cos k \quad k \in]0, \pi[. \tag{28}$$

Moreover one finds

$$\begin{aligned} \beta^{|x|-1} {}_2F_1\left(\frac{1 - |x|}{2}, 1 - \frac{|x|}{2}; 1 - |x|; \beta^{-2}\right) \\ = 2^{1-|x|} |x| {}_2F_1\left(1 - |x|, 1 + |x|; \frac{3}{2}; \frac{1 - \beta}{2}\right) \quad |x| \geq 1. \end{aligned} \tag{29}$$

This corresponds to the polynomial solutions (22) and (24), which have to match since, as polynomials, they are defined on the whole complex β -plane. (An amusing byproduct of the

foregoing analysis is the identity $1 = \sum_{n=1}^{\infty} \frac{(2n+1)!}{2^{3n}(2n-1)(n!)^2}$. The expansion around $\beta = -1$ yields similar results as can be expected from parity. This symmetry is not explicit on the series representations because the expansion point is not $\beta = 0$.

The limits $\beta \rightarrow \pm 1$ for the Green function do not exist. However, the following well-defined limit

$$G^{(1)}(x) \equiv \frac{1}{2} \lim_{\beta \rightarrow 1^-} (G_+^{(1)}(x|\beta) + G_-^{(1)}(x|\beta)) = \frac{1}{2}|x| \tag{30}$$

is also a solution to the Green equation. In fact the latter equation is, for any β , a one-dimensional recurrence relation which can be solved directly by the standard method. The points $\beta = \pm 1$ are degeneracy points and the direct solution in this case is $f(x) = (\pm 1)^x (f(0) \pm \frac{1}{2}|x|)$, for $\beta = \pm 1$, and $f(0)$ arbitrary. For $\beta = 1$, and with the choice $f(0) = 0$, one recovers (30).

4. The two-dimensional Green function

For arbitrary anisotropies, the identity (82) for the product of two Bessel functions can be used to obtain the following series expansion for the Green function:

$$G_{\pm}^{(2)}(\vec{x}|\vec{\alpha}, \beta) = -\frac{1}{|x_2|! 2\beta^{X+1}} \cdot \left(\frac{\alpha_1}{2}\right)^{|x_1|} \left(\frac{\alpha_2}{2}\right)^{|x_2|} \times \sum_{k=0}^{\infty} \frac{(X+2k)!}{k!(|x_1|+k)!} {}_2F_1\left(-k, -|x_1|-k; |x_2|+1; \left(\frac{\alpha_2}{\alpha_1}\right)^2\right) \cdot \left(\frac{\alpha_1}{2\beta}\right)^{2k} \tag{31}$$

When $\alpha_1 = \alpha_2$, these anisotropies can be set to 1 and the preceding expression becomes

$$G_{\pm}^{(2)}(\vec{x}|\beta) = -\frac{1}{(2\beta)^{X+1}} \sum_{k=0}^{\infty} \frac{(X+k+1)_k (X+2k)!}{k!(|x_1|+k)! (|x_2|+k)!} (2\beta)^{-2k} \tag{32}$$

$$= -\frac{X!}{(2\beta)^{X+1} |x_1|! |x_2|!} \times {}_4F_3\left(\frac{X+1}{2}, \frac{X+1}{2}, \frac{X}{2}+1, \frac{X}{2}+1; X+1, |x_1|+1, |x_2|+1; 4\beta^{-2}\right) \tag{33}$$

The series (32) converges uniformly for $|\beta| > 2$, simply for $|\beta| = 2$ and $\beta \neq \pm 2$ and diverges at $\beta = \pm 2$ and $|\beta| < 2$.

The recurrence relation for the coefficients of the Green function read

$$(n^2 - X^2)(n^2 - x^2)c_n - 4n^2(n-1)^2c_{n-2} = 0 \tag{34}$$

where $x \equiv |x_1| - |x_2|$. The Green function $G_{\pm}^{(2)}$ satisfies a fourth-order differential equation in β :

$$\beta^2(\beta^2 - 4)y'''' + \beta(10\beta^2 - 16)y''' + (\beta^2(-2(x_1^2 + x_2^2) + 25) - 8)y'' + 3\beta(-2(x_1^2 + x_2^2) + 5)y' + (1 - X^2)(1 - x^2)y = 0. \tag{35}$$

The four regular singular points and their indices are

$$\begin{aligned} \beta = 0: & \quad s = 1, 1, 0, 0 \\ \beta = \pm 2: & \quad s = 2, 1, 0, 0 \\ \beta = \infty: & \quad s = 1 + X, 1 - X, 1 + x, 1 - x. \end{aligned}$$

The three finite singular points were predicted in section 2. The series (32) is the regular solution corresponding to $(\beta = \infty; s = 1 + X)$. The degeneracy of the indices at the other points signals the existence of logarithmic solutions.

The radius of convergence of the series expansion around $\beta = 0$ is equal to 2, as $\beta = \pm 2$ are the closest singularities. Thus, this expansion and the expansion around infinity cover the whole complex β -plane. The recurrence relation for the coefficients of the series expansions around $\beta = 0$ contains two terms, just like (34) for the expansion around infinity. This permits the explicit determination of the series. One also has to find the connection coefficients involved in the linear combinations of the four solutions which reproduce the Green functions at hand, or the analytic continuations of (32). In the previous section, in (28), one had $\frac{|x|}{2}$ and $\pm \frac{1}{2\sqrt{2}}$. But the determination of the appropriate functions of \vec{x} is not an easy task and there is no general systematic method which gives a closed form result. Here it is possible to carry out this analysis completely, and the following expansions and connection coefficients have been found:

$$G_{\pm}^{(2)}(\vec{x}|\beta) = [f_0(\vec{x}) \pm l_0(\vec{x})]y_0(\beta) + [f_1(\vec{x}) \pm l_1(\vec{x})]y_1(\beta) \pm h_0(\vec{x})y_{l_0}(\beta) \pm h_1(\vec{x})y_{l_1}(\beta) \quad (36)$$

where

$$y_0(\beta) = {}_4F_3\left(\frac{1+X}{2}, \frac{1-X}{2}, \frac{1+x}{2}, \frac{1-x}{2}; 1, \frac{1}{2}, \frac{1}{2}; \frac{\beta^2}{4}\right) \quad (37)$$

$$y_1(\beta) = \beta {}_4F_3\left(\frac{2+X}{2}, \frac{2-X}{2}, \frac{2+x}{2}, \frac{2-x}{2}; \frac{3}{2}, \frac{3}{2}, 1; \frac{\beta^2}{4}\right) \quad (38)$$

$$y_{l_0}(\beta) = y_0(\beta) \ln \beta + \frac{d}{ds} {}_5F_4\left(\frac{s+1+X}{2}, \frac{s+1-X}{2}, \frac{s+1+x}{2}, \frac{s+1-x}{2}, 1; \frac{s+2}{2}, \frac{s+2}{2}, \frac{s+1}{2}, \frac{s+1}{2}; \frac{\beta^2}{4}\right)_{|s=0} \quad (39)$$

$$y_{l_1}(\beta) = y_1(\beta) \ln \beta + \beta \frac{d}{ds} {}_5F_4\left(\frac{s+1+X}{2}, \frac{s+1-X}{2}, \frac{s+1+x}{2}, \frac{s+1-x}{2}, 1; \frac{s+2}{2}, \frac{s+2}{2}, \frac{s+1}{2}, \frac{s+1}{2}; \frac{\beta^2}{4}\right)_{|s=1} \quad (40)$$

are the four independent solutions of the differential equation around $\beta = 0$. The s -derivatives can be easily expressed in terms of $\psi(z)$, the logarithmic derivative of the Gamma function. The logarithms are taken real for positive β . The six connection functions are given by

$$f_0(\vec{x}) = \frac{1}{4} \cos\left(\frac{\pi}{2}X\right) \cos\left(\frac{\pi}{2}x\right) + \frac{1}{4} \sin\left(\frac{\pi}{2}X\right) \sin\left(\frac{\pi}{2}|x|\right) \quad (41)$$

$$f_1(\vec{x}) = \frac{1}{2}|x_1^2 - x_2^2|f_0(\vec{x}) = \frac{1}{2}|x|Xf_0(\vec{x}) \quad (42)$$

$$h_0(\vec{x}) = h_0 \cos\left(\frac{\pi}{2}X\right) \cos\left(\frac{\pi}{2}x\right) \quad (43)$$

$$h_1(\vec{x}) = \frac{h_0}{2}|x_1^2 - x_2^2| \sin\left(\frac{\pi}{2}X\right) \sin\left(\frac{\pi}{2}|x|\right) = \frac{h_0}{2}xX \sin\left(\frac{\pi}{2}X\right) \sin\left(\frac{\pi}{2}x\right) \quad (44)$$

$$l_0(\vec{x}) = \left(\psi\left(\frac{X+1}{2}\right) + \psi\left(\frac{|x|+1}{2}\right)\right) h_0(\vec{x}) \quad (45)$$

$$l_1(\vec{x}) = \left(\psi\left(\frac{X}{2}+1\right) + \psi\left(\frac{|x|}{2}+1\right) - 2\right) h_1(\vec{x}) - \frac{h_0}{2}(|x|+X) \sin\left(\frac{\pi}{2}X\right) \sin\left(\frac{\pi}{2}|x|\right) \quad (46)$$

where $h_0 = \frac{i}{2\pi}$. These functions satisfy a number of equations of which the simplest are

$$Hf_0(\vec{x}) = \delta_{\vec{x},0} \quad Hh_0(\vec{x}) = 0 \quad Hl_0(\vec{x}) = 0 \quad (47)$$

$$Hf_1(\vec{x}) = 2f_0(\vec{x}) \quad Hh_1(\vec{x}) = 2h_0(\vec{x}) \quad Hl_1(\vec{x}) = 2l_0(\vec{x}) \quad (48)$$

$$x_1[f_0(x_1+1, x_2) - f_0(x_1-1, x_2)] = x_2[f_0(x_1, x_2+1) - f_0(x_1, x_2-1)]. \quad (49)$$

Thus f_0 is a Green function at the decoupling point while h_0 and l_0 are solutions of the homogeneous equation at the same point. The functions f_1, h_1 and l_1 appear as potentials for the sources f_0, h_0 and l_0 , respectively. The last equation for f_0 is a strong form of a directional-independence relation. The $f_0 - f_1$ part in (36) is a solution, regular at $\beta = 0$, of the Green equation. The remaining part is a singular solution of the homogeneous equation, independent from the value of h_0 . Finally, $h_0(\vec{x})y_0(\beta) + h_1(\vec{x})y_1(\beta)$ is a regular solution of the homogeneous equation.

The parity property is satisfied with $\ln(-1) = \pm i\pi$ and the complex conjugation symmetry for both β and β^* not on the branch cut.

We consider now the expansion about $\beta = 2$, and let $v = \beta - 2$. The index s can take one of the three values already found: 0, 1 or 2. A generic solution is given by

$$y(v) = \sum_{n \geq 0} a_n(s)v^{n+s} \tag{50}$$

where

$$\begin{aligned} &16(n+s)^2(n+s-1)(n+s-2)a_n \\ &+4(n+s-1)(n+s-2)[5n^2+(10s-9)n+5+5s^2-9s-2(x_1^2+x_2^2)]a_{n-1} \\ &+2(2n+2s-3)(n+s-2)[2n^2+(4s-6)n+5+2s^2 \\ &-6s-2(x_1^2+x_2^2)]a_{n-2}+(n+s-2+X)(n+s-2-X)(n+s-2+x) \\ &\times(n+s-2-x)a_{n-3} = 0 \end{aligned} \tag{51}$$

with $n \geq 0$ and $a_{<0} \equiv 0$. For $a_0(s) \equiv 1$, the solutions

$$y_2(v) = \sum_{n \geq 0} a_n(s=2)v^{n+2} \tag{52}$$

$$y_1(v) = \sum_{n \geq 0} a_n(s=1)v^{n+1} \quad a_1 \equiv 0 \tag{53}$$

$$y_0(v) = \sum_{n \geq 0} a_n(s=0)v^n \quad a_1 = a_2 \equiv 0 \tag{54}$$

are linearly independent and regular at $\beta = 2$. The fourth solution has the expected logarithmic singularity

$$y_{l0}(v) = y_0(v) \ln v + \sum_{n \geq 0} v^n \frac{d}{ds} a_n(s)|_{s=0}. \tag{55}$$

The complete Green function is *a priori* a linear combination of the four solutions. The corresponding connection coefficients have not been investigated. However, one can conclude that the logarithmic solution corresponds to a solution of the homogeneous equation (3), and can therefore be dropped without altering the Green property. The remaining piece is the natural regularization at $\beta = 2$ of the Green function defined by (5). The situation at $\beta = -2$ is similar.

For $x_1 = x_2$ and arbitrary anisotropies, Montroll found an expression in terms of Legendre functions [17]. In terms of the definitions adopted here one has

$$G_M^{(2)}((x_1, x_1)|(\alpha_1, \alpha_2), \beta) = -\frac{1}{2\pi(\alpha_1\alpha_2)^{\frac{1}{2}}} Q_{|x_1|-\frac{1}{2}}\left(\frac{\beta^2 - (\alpha_1^2 + \alpha_2^2)}{2\alpha_1\alpha_2}\right) \tag{56}$$

where $Q_\nu(z)$ is the ν th Legendre function of the second kind. These functions have logarithmic singularities at $z = \pm 1$. But for ν a half odd-integer, only $z = 1$ is a singularity:

$$Q_{|x_1|-\frac{1}{2}}(1-\epsilon) \sim -\frac{1}{2} \ln\left(\frac{\epsilon}{2}\right) - \gamma - \psi\left(|x_1| + \frac{1}{2}\right) + O(\epsilon) \quad \epsilon \rightarrow 0^+ \tag{57}$$

where $\gamma = 0.577\,216\dots$. Thus, one finds a logarithmic divergence at $\beta = \pm 2$:

$$G_M^{(2)}((x_1, x_1)|(1, 1), \beta) = \frac{1}{4\pi} \ln\left(\frac{4 - \beta^2}{4}\right) + \frac{\gamma}{2\pi} + \frac{1}{2\pi} \psi\left(|x_1| + \frac{1}{2}\right) + O\left(\frac{\beta^2 - 4}{2}\right) \quad \beta \rightarrow 2^- \quad \text{or} \quad \beta \rightarrow -2^+. \tag{58}$$

One also has $G_M^{(2)}((x_1, x_1)|(1, 1), 0) = -\frac{1}{4}$.

Montroll’s result should be qualified. It is general in terms of anisotropies, but partial as it applies only to the line $x_1 = x_2$ and $|\beta| < 2$. Also, it is only a part of the full Green function, even under these restrictions. Consider $x_1 = x_2$ in the Green function given by (36). At $\beta = 0$ this expression diverges, as expected, from the analysis of section 2, and thus contradicts (56). However, taking the half-sum of the two Green functions one finds

$$G^{(2)}(\vec{x}|\beta) = \frac{1}{2}(G_+^{(2)}(\vec{x}|\beta) + G_-^{(2)}(\vec{x}|\beta)) = f_0(\vec{x})y_0(\beta) + f_1(\vec{x})y_1(\beta). \tag{59}$$

This solution of the Green equation is regular at $\beta = 0$, and

$$G^{(2)}((x_1, x_1)|\beta) = \frac{1}{4} \cos(\pi x_1) {}_2F_1\left(\frac{1}{2} + |x_1|, \frac{1}{2} - |x_1|; 1; \frac{\beta^2}{4}\right). \tag{60}$$

For $x_1 = 0$, one has the following identity:

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\beta^2}{4}\right) = \frac{2}{\pi} K\left(\frac{\beta^2}{4}\right) \tag{61}$$

where K is the complete elliptic function of the first kind [18]. One also has

$$Q_{|x_1|-\frac{1}{2}}\left(\frac{\beta^2 - 2}{2}\right) = \frac{\pi}{2} \cos(\pi x_1) {}_2F_1\left(\frac{1}{2} + |x_1|, \frac{1}{2} - |x_1|; 1; \frac{\beta^2}{4}\right) \quad |\beta| < 2 \tag{62}$$

which shows that $G^{(2)}((x_1, x_1)|\beta)$ and $G_M^{(2)}((x_1, x_1)|(1, 1), \beta)$ are equal, up to a factor of -1 . The origin of this sign in (56) is unclear. We now compare (56) to the half-sum of the two functions (32) at $x_1 = x_2$. The former function is even in β while the latter is odd. Therefore, they cannot be equal, and (56) holds only for $|\beta| < 2$.

5. The three-dimensional Green function

Using the identity (82) for the product of two Bessel functions one obtains the following series expansion for the three-dimensional function:

$$G_{\pm}^{(3)}(\vec{x}|\vec{\alpha}, \beta) = -\frac{1}{|x_2|!2\beta^{X+1}} \cdot \left(\frac{\alpha_1}{2}\right)^{|x_1|} \left(\frac{\alpha_2}{2}\right)^{|x_2|} \left(\frac{\alpha_3}{2}\right)^{|x_3|} \times \sum_{k=0}^{\infty} (X + 2k)! \left(\frac{\alpha_3}{2\beta}\right)^{2k} \sum_{p=0}^k \frac{1}{p!(k-p)! (|x_1| + p)! (|x_3| + k - p)!} \times \left(\frac{\alpha_1}{\alpha_3}\right)^{2p} {}_2F_1\left(-p, -|x_1| - p; |x_2| + 1; \left(\frac{\alpha_2}{\alpha_1}\right)^2\right). \tag{63}$$

When the anisotropies α_j are equal to 1 the preceding expression simplifies to

$$G_{\pm}^{(3)}(\vec{x}|\beta) = -\frac{1}{(2\beta)^{X+1}} \sum_{k=0}^{\infty} \frac{(X + 2k)!}{(2\beta)^{2k}} \times \sum_{p=0}^k \frac{(|x_1| + |x_2| + p + 1)_p}{p!(k-p)! (|x_1| + p)! (|x_2| + p)! (|x_3| + k - p)!}. \tag{64}$$

The series (64) converges for $|\beta| \geq 3$. For $\vec{x} = \vec{0}$ one can write

$$G_{\pm}^{(3)}(\vec{0}|\beta) = -\frac{1}{2\beta} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} u_k \beta^{-2k} \tag{65}$$

$$u_k = \sum_{p=0}^k \binom{k}{p}^2 \binom{2p}{p}. \tag{66}$$

The value of this series at $\beta = 3$ was calculated by Watson [19]:

$$G_{\pm}^{(3)}(\vec{0}|3) = -\frac{2}{\pi^2} [18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}] K^2 ((2 - \sqrt{3})^2 (\sqrt{3} - \sqrt{2})^2). \tag{67}$$

The numerical value of (67) is $-0.2527\dots$ (The definition of [18] is adopted for the function K .) At the singular point $\beta = 3$, this series converges, rather slowly, to the known value (67). The sum of the first 1001 terms gives $-0.2502\dots$. It is amusing to note that the u_k ($k \geq 1$) appear to be divisible by 3, the number of dimensions.

It is possible to derive a recurrence relation for the coefficients of the three-dimensional Green function. We define the following even homogeneous polynomials:

$$\begin{aligned} \Sigma_{222} &= x_1^2 x_2^2 x_3^2 \\ \Sigma_{422} &= x_1^2 x_2^2 x_3^2 (x_1^2 + x_2^2 + x_3^2) \\ \Sigma_{2i} &= x_1^{2i} + x_2^{2i} + x_3^{2i} \quad i = 1, 2, 3, 4 \\ \Sigma_{(2i)(2i)} &= x_1^{2i} x_2^{2i} + x_2^{2i} x_3^{2i} + x_1^{2i} x_3^{2i} \quad i = 1, 2 \\ \Sigma_{(2i)2} &= x_1^{2i} x_2^2 + x_1^2 x_2^{2i} + x_2^{2i} x_3^2 + x_2^2 x_3^{2i} + x_1^{2i} x_3^2 + x_1^2 x_3^{2i} \quad i = 2, 3. \end{aligned}$$

For $c_n \equiv c_n^{(3)}(\vec{x}|(1, 1, 1))$, it has been found that

$$\begin{aligned} (n-2)(n-4)[\Sigma_8 - 4\Sigma_{62} + 6\Sigma_{44} + 4\Sigma_{422} \\ - 4n^2(\Sigma_6 - \Sigma_{42} + 10\Sigma_{222}) + 2n^4(3\Sigma_4 + 2\Sigma_{22}) - 4n^6\Sigma_2 + n^8]c_n \\ + 4n(n-1)^2(n-4)[- \Sigma_6 + \Sigma_{42} \\ + 6\Sigma_{222} - (n^2 - 4n - 2)\Sigma_4 - 2(3n^2 - 4n + 2)\Sigma_{22} \\ + n(n-2)(5n^2 - 6n + 8)\Sigma_2 - n(n-2)(n^2 + 2)(3n^2 - 6n + 4)]c_{n-2} \\ + 2n(n-1)(n-2)(n-3)[-(n^2 - 4n + 12)\Sigma_4 + 2(5n^2 - 20n + 12)\Sigma_{22} \\ - 2n(n-4)(7n^2 - 24n + 28)\Sigma_2 \\ + n(n-4)(15n^4 - 96n^3 + 268n^2 - 384n + 248)]c_{n-4} \\ + 4n^2(n-1)(n-2)(n-3)(n-4)(n-5) \\ \times [3(n-3)\Sigma_2 - (7n^3 - 57n^2 + 158n - 162)]c_{n-6} \\ + 9n^2(n-1)(n-2)^2(n-3)(n-4)(n-5)(n-6)(n-7)c_{n-8} = 0. \tag{68} \end{aligned}$$

This translates into a tenth-order differential equation for $y = G_{\pm}^{(3)}$:

$$\begin{aligned} \beta^2(\beta^2 - 1)^3(\beta^2 - 9)y^{(10)} + \beta(\beta^2 - 1)^2(61\beta^4 - 418\beta^2 + 45)y^{(9)} \\ - (\beta^2 - 1)[\beta^6(4\Sigma_2 - 1433) \\ + \beta^4(-16\Sigma_2 + 7511) + \beta^2(12\Sigma_2 - 2673) + 27]y^{(8)} \\ - 4\beta[\beta^6(42\Sigma_2 - 4167) + \beta^4(-146\Sigma_2 + 17284) \\ + \beta^2(122\Sigma_2 - 11683) - 18\Sigma_2 + 1140]y^{(7)} \\ + [\beta^6(6\Sigma_4 + 4\Sigma_{22} - 2552\Sigma_2 + 102963) \\ + \beta^4(-4\Sigma_4 - 24\Sigma_{22} + 5740\Sigma_2 - 261972) \end{aligned}$$

$$\begin{aligned}
& +\beta^2(-2\Sigma_4 + 20\Sigma_{22} - 2444\Sigma_2 + 90\,750) + 48\Sigma_2 - 1536]y^{(6)} \\
& -\beta[\beta^4(-162\Sigma_4 - 108\Sigma_{22} + 17\,640\Sigma_2 - 337\,617) \\
& +\beta^2(76\Sigma_4 + 392\Sigma_{22} - 23\,180\Sigma_2 + 470\,364) + 14\Sigma_4 - 140\Sigma_{22} \\
& +3812\Sigma_2 - 64\,194]y^{(5)} \\
& -2[\beta^4(2\Sigma_6 - 2\Sigma_{42} + 20\Sigma_{222} - 714\Sigma_4 - 476\Sigma_{22} + 28\,742\Sigma_2 - 278\,744) \\
& +\beta^2(2\Sigma_6 - 2\Sigma_{42} - 12\Sigma_{222} + 202\Sigma_4 + 892\Sigma_{22} \\
& -18\,870\Sigma_2 + 179\,166) + 17\Sigma_4 - 74\Sigma_{22} + 590\Sigma_2 - 5055]y^{(4)} \\
& -4\beta[\beta^2(16\Sigma_6 - 16\Sigma_{42} + 160\Sigma_{222} - 1230\Sigma_4 - 820\Sigma_{22} + 20\,776\Sigma_2 \\
& -103\,135) + 8\Sigma_6 - 8\Sigma_{42} - 48\Sigma_{222} + 150\Sigma_4 + 612\Sigma_{22} - 5190\Sigma_2 + 23\,022]2y^{(3)} \\
& +[\beta^2(\Sigma_8 - 4\Sigma_{62} + 6\Sigma_{44} + 4\Sigma_{422} - 276\Sigma_6 + 276\Sigma_{42} - 2760\Sigma_{222} \\
& +6246\Sigma_4 + 4164\Sigma_{22} - 44\,916\Sigma_2 + 108\,681) \\
& -40\Sigma_6 + 40\Sigma_{42} + 240\Sigma_{222} - 120\Sigma_4 - 720\Sigma_{22} + 2280\Sigma_2 - 4680]y^{(2)} \\
& +9\beta[\Sigma_8 - 4\Sigma_{62} + 6\Sigma_{44} + 4\Sigma_{422} - 36\Sigma_6 \\
& +36\Sigma_{42} - 360\Sigma_{222} + 246\Sigma_4 + 164\Sigma_{22} - 676\Sigma_2 + 681]y^{(1)} \\
& +15[\Sigma_8 - 4\Sigma_{62} + 6\Sigma_{44} + 4\Sigma_{422} \\
& -4\Sigma_6 + 4\Sigma_{42} - 40\Sigma_{222} + 6\Sigma_4 + 4\Sigma_{22} - 4\Sigma_2 + 1]y = 0. \tag{69}
\end{aligned}$$

This equation has six regular singular points, $0, \pm 1, \pm 3$ and ∞ . The corresponding indices are

$$\begin{aligned}
\beta = 0: & \quad s = 7, 7, 6, 5, 5, 4, 3, 2, 1, 0 \\
\beta = \pm 1: & \quad s = 6, 5, 4, 3, 2, 1, 0, \frac{5}{2}, \frac{3}{2}, \frac{1}{2} \\
\beta = \pm 3: & \quad s = 8, 7, 6, 5, 4, 3, 2, 1, 0, \frac{1}{2} \\
\beta = \infty: & \quad s = 5, 3, 1 + |x_1| + |x_2| + |x_3|, 1 - |x_1| - |x_2| - |x_3|, \\
& \quad 1 + |x_1| + |x_2| - |x_3|, 1 + |x_1| - |x_2| + |x_3|, 1 - |x_1| + |x_2| + |x_3|, \\
& \quad 1 + |x_1| - |x_2| - |x_3|, 1 - |x_1| - |x_2| + |x_3|, 1 - |x_1| + |x_2| - |x_3|.
\end{aligned}$$

The Green function is regular at $\beta = \infty$ and corresponds to the index $1 + |x_1| + |x_2| + |x_3|$. The appearance of indices differing by integer values can result in logarithms. However, the ten-term recurrence for the series expansion around $\beta = \pm 3$ shows that all ten solutions do not contain logarithms. The same conclusion holds at $\beta = \pm 1$, with an eight-term recurrence. Therefore $\beta = \pm 1$ and $\beta = \pm 3$ are branch point singularities which are free of logarithms. At $\beta = 0$ the indices indicate that some solutions contain logarithms. For the foregoing Green function $\beta = 0$ is a regular point, and one should consider a linear combination of the logarithm-free solutions.

The reason for the appearance of $\beta = 0$ as a singularity of the differential equation is unclear. Perhaps the fact that this point is a fixed point of the parity symmetry, or its status as a decoupling point may be relevant here.

At its four singular points, the Green function does not diverge as all the indices are non-negative; only the solutions which correspond to the vanishing indices give non-vanishing contributions. However, around a given singular point, one can still consider a natural regularization by dropping all the solutions in the corresponding linear combination which are associated with non-integer indices. Such solutions combine into a solution of the homogeneous equation (3). Finally note that some results were obtained by Joyce [20], at $\vec{x} = \vec{0}$ and around $\beta = 3$.

6. Concluding remarks

Using equation (82) twice gives the large β series expansion of the four-dimensional Green function

$$\begin{aligned}
 G_{\pm}^{(4)}(\vec{x}|\vec{\alpha}, \beta) &= -\frac{\prod_{j=1}^4 \left(\frac{\alpha_j}{2}\right)^{|x_j|}}{|x_2|!|x_4|!} \cdot \frac{1}{2\beta^{X+1}} \sum_{k=0}^{\infty} (X+2k)! \left(\frac{\alpha_3}{2\beta}\right)^{2k} \\
 &\quad \times \sum_{p=0}^k \frac{1}{p!(k-p)! (|x_1|+p)! (|x_3|+k-p)!} \left(\frac{\alpha_1}{\alpha_3}\right)^{2p} \\
 &\quad \times {}_2F_1\left(-p, -|x_1|-p; |x_2|+1; \left(\frac{\alpha_2}{\alpha_1}\right)^2\right) \\
 &\quad \times {}_2F_1\left(-(k-p), -|x_3|-(k-p); |x_4|+1; \left(\frac{\alpha_4}{\alpha_3}\right)^2\right). \tag{70}
 \end{aligned}$$

When the anisotropies are all equal to 1 the preceding expression becomes

$$\begin{aligned}
 G_{\pm}^{(4)}(\vec{x}|\beta) &= -\frac{1}{(2\beta)^{(X+1)}} \sum_{k=0}^{\infty} \frac{(X+2k)!}{(2\beta)^{2k}} \\
 &\quad \times \sum_{p=0}^k \frac{(|x_1|+|x_2|+p+1)_p (|x_3|+|x_4|+k-p+1)_{k-p}}{p!(k-p)! (|x_1|+p)! (|x_2|+p)! (|x_3|+k-p)! (|x_4|+k-p)!}. \tag{71}
 \end{aligned}$$

The series (71) converges for $|\beta| \geq 4$. For $\vec{x} = \vec{0}$ one can write

$$G_{\pm}^{(4)}(\vec{0}|\beta) = -\frac{1}{2\beta} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} v_k \beta^{-2k} \tag{72}$$

$$v_k = \sum_{p=0}^k \binom{k}{p}^2 \binom{2k-2p}{k-p} \binom{2p}{p}. \tag{73}$$

For $\beta = 4$, this series converges to the known value of $-0.1549\dots$ [12]. The sum of the first 31 terms gives $-0.1541\dots$. Similarly to u_k in the preceding section, v_k ($k \geq 1$) also appear to be divisible by the number of dimensions, here equal to four.

A derivation of the recurrence relation for the coefficients $c_n^{(4)}(\vec{x})$ can be done as for the lower dimensions. The order of the recurrence appears to be larger than seven. From this recurrence a differential equation can be derived. The five singular points are of the logarithmic type. The logarithmic solutions, at one given singular point, can be dropped leaving a regular Green function.

These features were seen to be common to the lowest dimensions. They also hold for all the higher dimensions. The coefficients $c_n^{(d)}(\vec{x})$ satisfy recurrence relations for all dimensions. The general form of these relations is easily inferred from the results for the lower dimensions. The coefficients appearing in the recurrence relations are polynomials in n and the x_j^2 . From such relations one can then derive the differential equation as was done for the lowest dimensions. Another common feature is the possibility of dropping the singular part, around one given singularity. This part is a solution of the homogeneous equation. Finally, at $\beta = d$, subtracting $G_{\pm}^{(d)}(\vec{0}|d)$ from (5) provides another regularization (see [13] for $d = 2$).

Determining the explicit recurrence relation and the differential equation is however not a trivial task. It is also difficult to find the \vec{x} -dependent coefficients appearing in the linear combination of the solutions around a given singularity. These techniques were applied for

the lower dimensions and new results were obtained. While the low dimensions studied in this paper seem to be at the limit of tractability of these methods, the knowledge obtained about the analytic structure of the lattice Green functions in all dimensions is an important step in their study.

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Appendix A. Bessel functions and other formulae

The cylindrical Bessel functions $J_n(z)$ have both integral representations

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi dq \exp(iz \cos q) \cos(nq) \quad \forall n \in \mathbb{Z} \quad (74)$$

and series expansions

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{z}{2}\right)^{2k} \quad (75)$$

with an infinite radius of convergence. The asymptotic behaviour at infinity is given by

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \left(\cos \chi - \frac{4n^2 - 1}{8z} \sin \chi \right) \quad |z| \rightarrow \infty \quad |\arg(z)| < \pi \quad (76)$$

where $\chi = z - \frac{\pi}{2}n - \frac{\pi}{4}$. One also has

$$\int_0^\infty J_n(x) dx = 1 \quad n \geq 0. \quad (77)$$

The Bessel functions have the following properties:

$$J_n(-z) = (-1)^n J_n(z) \quad \forall n \in \mathbb{Z} \quad (78)$$

$$J_{-n}(z) = (-1)^n J_n(z) \quad \forall n \in \mathbb{Z} \quad (79)$$

$$J_0(0) = +1 \quad J_n(0) = 0 \quad n \neq 0 \quad (80)$$

$$J_{n-1}(z) - J_{n+1}(z) = 2J'_n(z) \quad \forall n \in \mathbb{Z}. \quad (81)$$

A particular formula for the product of two Bessel functions is

$$J_m(az)J_n(bz) = \frac{\left(\frac{az}{2}\right)^m \left(\frac{bz}{2}\right)^n}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-1)^k {}_2F_1(-k, -m-k; n+1; \frac{b^2}{a^2})}{k! \Gamma(m+k+1)} \left(\frac{az}{2}\right)^{2k}. \quad (82)$$

(A typographical error in [21] has been corrected.) Note that ${}_2F_1$ is in fact a polynomial in b^2/a^2 . When $a = b = 1$ this formula simplifies to

$$J_m(z)J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (m+n+k+1)_k}{k! \Gamma(m+k+1) \Gamma(n+k+1)} \left(\frac{z}{2}\right)^{m+n+2k}. \quad (83)$$

The generalized hypergeometric series ${}_pF_q$ are defined by [16]

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \cdot \frac{z^k}{k!} \tag{84}$$

where the Pochhammer symbol $(a)_k$ is defined by

$$(a)_0 = 1 \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1) \dots (a+k-1) \tag{85}$$

and $p \leq q + 1$. We define a differential operator $\delta = z \frac{d}{dz}$. The function (84) satisfies the differential equation

$$[\delta(\delta + b_1 - 1) \dots (\delta + b_q - 1) - z(\delta + a_1) \dots (\delta + a_p)]y(z) = 0. \tag{86}$$

For $p = q + 1$ this equation is Fuchsian with three regular singular points at 0, 1 and ∞ .

Appendix B. The WLW approach for arbitrary mass

Starting from an observation of C Vohwinkel, Lüscher and Weisz have developed a powerful algorithmic method for the numerical calculation of the massless lattice Green function in four dimensions [12]. Their method applies immediately to any dimension. Here it is shown that this method generalizes to arbitrary mass and anisotropies. A conserved quantity for the two-dimensional Green function at the massless and decoupling points is also derived.

An integration by parts of the left-hand side of (87) yields the right-hand side:

$$\alpha_j (G_{\pm}^{(d)}(\vec{x} + \hat{e}_j | \vec{\alpha}, \beta) - G_{\pm}^{(d)}(\vec{x} - \hat{e}_j | \vec{\alpha}, \beta)) = -x_j \mathcal{H}(\vec{x} | \vec{\alpha}) \quad j = 1, \dots, d \tag{87}$$

$$\mathcal{H}(\vec{x} | \vec{\alpha}) = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \frac{d^d \vec{q}}{(2\pi)^d} \exp(i\vec{q} \cdot \vec{x}) \ln \left(2\beta - 2 \sum_{j=1}^d \alpha_j \cos q_j \mp i\epsilon \right). \tag{88}$$

Equation (1) allows one to find another expression for \mathcal{H} :

$$\mathcal{H}(\vec{x} | \vec{\alpha}) = \frac{2}{\sum_{j=1}^d x_j} \left(\sum_{j=1}^d \alpha_j G_{\pm}^{(d)}(\vec{x} - \hat{e}_j | \vec{\alpha}, \beta) - \beta G_{\pm}^{(d)}(\vec{x} | \vec{\alpha}, \beta) \right) \tag{89}$$

provided $\sum_{j=1}^d x_j \neq 0$. This gives the value of $G_{\pm}^{(d)}(\vec{x} + \hat{e}_j | \vec{\alpha}, \beta)$ in terms of $G_{\pm}^{(d)}(\vec{x} | \vec{\alpha}, \beta)$ and $G_{\pm}^{(d)}(\vec{x} - \hat{e}_k | \vec{\alpha}, \beta)$. The repeated use of these recurrence relations, coupled with the $\pm x_j$ invariance, shows that $G_{\pm}^{(d)}(\vec{x} | \vec{\alpha}, \beta)$ is a linear combination of the 2^d values corresponding to $x_j = 0, 1$. Note that all the vertices of the unit hypercube are needed when the anisotropies are arbitrary. These 2^d values can be calculated numerically, and the particular ‘ \pm ’ branch obtained, depending on the given value of β in the complex plane. This generalizes the approach developed in [12].

One can look for additional conserved quantities as was done in [12]. However, this method depends rather strongly on the dimension. For the isotropic two-dimensional case, we define

$$g_0(n) = G_{\pm}^{(2)}((n, 0) | \beta) \quad g_1(n) = G_{\pm}^{(2)}((n, 1) | \beta) \quad n \geq 0. \tag{90}$$

The Green equation gives

$$g_0(n+1) + g_0(n-1) + 2g_1(n) - 2\beta g_0(n) = 0 \quad n \geq 1 \tag{91}$$

and an equation inferred from the above approach is

$$g_1(n+1) = \frac{2n}{n+1} (\beta g_1(n) - g_0(n)) - \frac{n-1}{n+1} g_1(n-1) \quad n \geq 1. \tag{92}$$

One can look for a conserved quantity in the following form:

$$C(n) = ng_0(n) + a_1ng_1(n) + b_0(n-1)g_0(n-1) + b_1(n-1)g_1(n-1) \\ + c_0g_0(n) + c_1g_1(n) + d_0g_0(n-1) + d_1g_1(n-1) \quad n \geq 1. \quad (93)$$

Using (91) and (92) one finds that $C(n)$ is independent of n provided

$$b_0 = c_0 = d_0 = -1 \quad a_1 = -b_1 = \frac{1}{\beta - 1} \quad c_1 = d_1 = 0 \quad (94)$$

and $\beta = 0$ or 2 . This form does not allow for other values of β , but a conserved quantity at $\beta = -2$ can be obtained from the one for $\beta = 2$ through the parity symmetry. The new quantity is *a priori* conserved for n odd and n even separately. It would be interesting to find out whether arbitrary values of β accommodate conserved quantities.

Note that $\beta = 0, \pm 2$ are the three singular values. Therefore $C(n)$ may not be well defined. However, using the explicit expression of section 4, and taking the limit $\beta \rightarrow 0$, one finds that the infinities cancel exactly, leaving $C_{\pm} \equiv C_{\pm}(n) = \pm \frac{i}{\pi}$ for all $n \geq 1$. For the half-sum the conserved quantity is therefore $C = 0$. The situation at $\beta = 2$ is similar. The conserved quantities can be finite through cancellations, and the divergence corresponds to a solution of the homogeneous equation and can therefore be dropped. (See also the remark in the conclusion of [12] concerning this conserved quantity, and [13].)

Note added in proof. Lattice Green functions also arise in the study of the statistical mechanics of the spherical model [22]. Complex temperature singularities of this system were studied in [23]. Lattice Green functions were also examined for the cases where factorizations in two complete elliptic integrals occur [24]. I would like to thank P Butera for bringing to my attention the four works cited in this paper.

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